EXAM IMAGE PROCESSING

15-04-2009

A FORMULA SHEET IS INCLUDED ON PAGES 3-4

Put your name on all pages which you hand in, and number them. Write the total number of pages you hand in on the first page. Write clearly and not with pencil or red pen. The use of a simple calculator (not a graphical one) is allowed. ***Always motivate your answers.*** *Good luck!*

Problem 1 (30 pt)
The general convolution filter of size 3×3 is defined by the filter mask in Figure 1. In the following questions, "Give an example" means "give values of $w_1, \ldots, w_9$".

w_1	w_2	w_3
w_4	w_5	w_6
w_7	w_8	w_9

FIGURE 1: *General* 3×3 *filter mask.*

a. Give an example for which the filter extracts horizontal edges.
b. Give an example for which the filter smooths the input image.
c. Give an example for which the filter computes a discrete Laplacian of the input image.
d. Give an example for which the result can be also achieved by two passes using 1-D masks, and explicitly indicate those 1-D masks.
e. Under what condition on $w_1, \ldots, w_9$ is the average grey value of the filtered image equal to the average grey value of the input image?
f. Under what condition on $w_1, \ldots, w_9$ is the average grey value of the filtered image equal to zero?

Problem 2 (30 pt)
Consider the binary image in Figure 2 (left), which contains a piece of text. The task is to produce an output image which contains all letters of the input image except those which touch the boundary. This can be done in two steps. Step 1 is to extract all letters which touch the boundary. For this purpose one can use a morphological reconstruction.

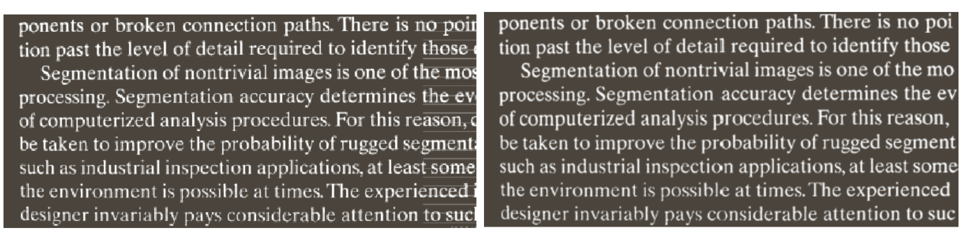

FIGURE 2: *Left: input image; Right: image with no objects touching the boundary.*

a. Explain how the marker image F, the mask image G and the structuring element B of the morphological reconstruction have to be chosen.
b. Explain how the iterative process of morphological reconstruction leads to the extraction of all letters which touch the boundary.
c. Give step 2 of the processing chain which leads to the desired output.
d. Describe two other image processing tasks involving morphological reconstruction.

(*continue on page 2*)

Problem 3 (30 pt)
Consider the textures in Figure 3.

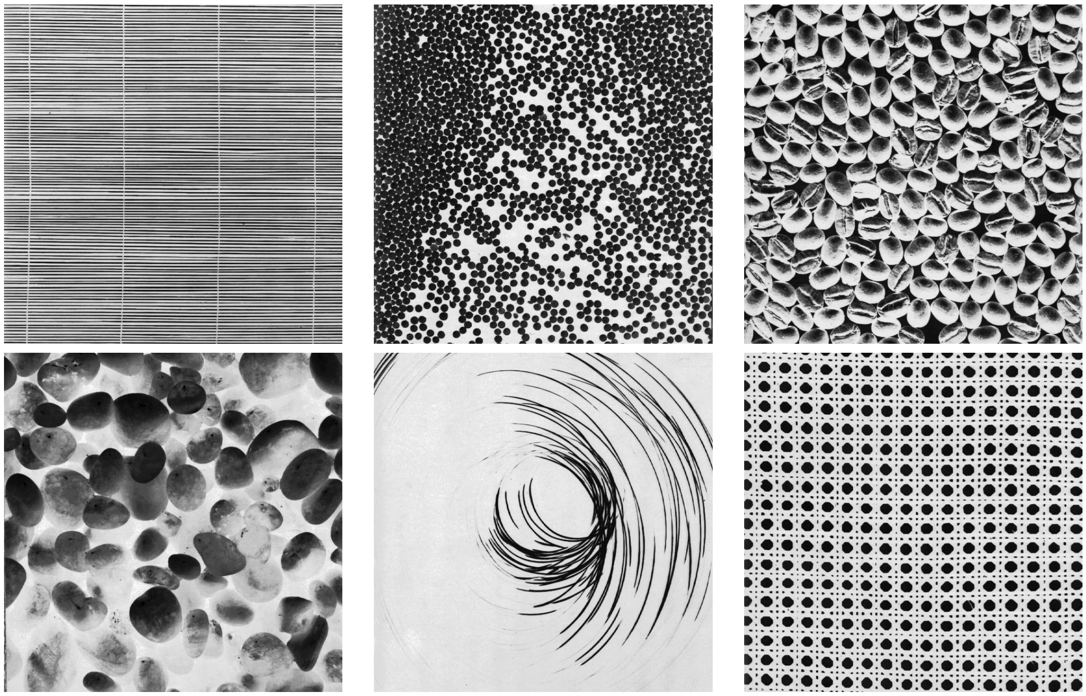

FIGURE 3: *Examples from the Brodatz texture database.*

We want to apply a set of image operations and descriptors to these images so that they can be distinguished. Many solutions are possible, and there is no absolutely "correct" solution. Give a set of image operations and descriptors of your choice (aim for a convincing motivation!). Clearly indicate:

a. A short explanation of each image operation / descriptor in your set.
b. How your set of operations / descriptors would allow to distinguish between the six textures.

(*Formula sheet on next page*)

Formula sheet

Co-occurrence matrix $g(i, j) = \{\text{no. of pixel pairs with grey levels } (z_i, z_j) \text{ satisfying predicate } Q\},\ 1 \leq i, j \leq L$

Convolution, 2-D discrete $(f \star h)(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)\, h(x - m, y - n)$, for $x = 0, 1, 2, \ldots, M - 1, y = 0, 1, 2, \ldots, N - 1$

Convolution Theorem, 2-D discrete $\mathcal{F}\{f \star h\}(u, v) = F(u, v)\, H(u, v)$

Distance measures Euclidean: $D_e(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$, City-block: $D_4(p, q) = |p_1 - q_1| + |p_2 - q_2|$, Chessboard: $D_8(p, q) = \max(|p_1 - q_1|, |p_2 - q_2|)$

Entropy, source $H = -\sum_{j=1}^{J} P(a_j) \log P(a_j)$

Entropy, estimated for L-level image: $\tilde{H} = -\sum_{k=0}^{L-1} p_r(r_k) \log_2\, p_r(r_k)$

Error, root-mean square $e_{\text{rms}} = \left[\frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left(\widehat{f}(x, y) - f(x, y) \right)^2 \right]^{\frac{1}{2}}$

Exponentials $e^{ix} = \cos x + i \sin x; \quad \cos x = (e^{ix} + e^{-ix})/2; \quad \sin x = (e^{ix} - e^{-ix})/2i$

Filter, inverse $\widehat{\mathbf{f}} = \mathbf{f} + \mathbf{H}^{-1}\mathbf{n}$, $\widehat{F}(u, v) = F(u, v) + \frac{N(u,v)}{H(u,v)}$

Filter, parametric Wiener $\widehat{\mathbf{f}} = \left(\mathbf{H}^t\mathbf{H} + K\,\mathbf{I}\right)^{-1} \mathbf{H}^t\mathbf{g}$, $\widehat{F}(u, v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + K} \right] G(u, v)$

Fourier series of signal with period T: $f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{i \frac{2\pi n}{T} t}$, with Fourier coefficients: $c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, e^{-i \frac{2\pi n}{T} t}\, dt, \quad n = 0, \pm 1, \pm 2, \ldots$

Fourier transform 1-D (continuous) $F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\mu t}\, dt$

Fourier transform 1-D, inverse (continuous) $f(t) = \int_{-\infty}^{\infty} F(\mu)\, e^{i2\pi\mu t}\, d\mu$

Fourier Transform, 2-D Discrete $F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)\, e^{-i2\pi(u\,x/M + v\,y/N)}$ for $u = 0, 1, 2, \ldots, M - 1, v = 0, 1, 2, \ldots, N - 1$

Fourier Transform, 2-D Inverse Discrete $f(x, y) = \frac{1}{M\,N} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v)\, e^{i2\pi(u\,x/M + v\,y/N)}$ for $x = 0, 1, 2, \ldots, M - 1, y = 0, 1, \ldots, N - 1$

Fourier spectrum Fourier transform of $f(x, y)$: $F(u, v) = R(u, v) + i\, I(u, v)$, Fourier spectrum: $|F(u, v)| = \sqrt{R^2(u, v) + I^2(u, v)}$, phase angle: $\phi(u, v) = \arctan\left(\frac{I(u,v)}{R(u,v)} \right)$

Gaussian function mean μ, variance σ^2: $G_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}$

Gradient $\nabla f(x, y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$

Histogram $h(m) = \#\{(x, y) \in D : f(x, y) = m\}$. Cumulative histogram: $\mathsf{P}(\ell) = \sum_{m=0}^{\ell} h(m)$

Impulse, discrete $\delta(0) = 1, \delta(x) = 0$ for $x \in \mathbb{N} \backslash \{0\}$

Impulse, continuous $\delta(0) = \infty, \delta(x) = 0$ for $x \neq 0$, with $\int_{-\infty}^{\infty} f(t)\, \delta(t - t_0)\, dt = f(t_0)$

Impulse train $s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$, with Fourier transform $S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$

Laplacian $\nabla^2 f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

Laplacian-of-Gaussian $\nabla^2 G_\sigma(x, y) = -\frac{2}{\pi\sigma^4} \left(1 - \frac{r^2}{2\sigma^2} \right) e^{-r^2/2\sigma^2} \quad (r^2 = x^2 + y^2)$

Median The median of an odd number of numerical values is found by arranging all the numbers from lowest value to highest value and picking the middle one.

Morphology

Dilation $\delta_A(X) = X \oplus A = \bigcup_{a\in A} X_a = \bigcup_{x\in X} A_x = \{h \in E : \check{A}_h \cap X \neq \emptyset\}$,
where $X_h = \{x + h : x \in X\}$, $h \in E$ and $\check{A} = \{-a : a \in A\}$

Erosion $\varepsilon_A(X) = X \ominus A = \bigcap_{a\in A} X_{-a} = \{h \in E : A_h \subseteq X\}$

Opening $\gamma_A(X) = X \circ A := (X \ominus A) \oplus A = \delta_A\varepsilon_A(X)$

Closing $\phi_A(X) = X \bullet A := (X \oplus A) \ominus A = \varepsilon_A\delta_A(X)$

Hit-or-miss transform $X \circledast (A_1, A_2) = (X \ominus A_1) \cap (X^c \ominus A_2)$

Thinning $X \otimes A = X \backslash (X \circledast A)$, **Thickening** $X \odot A = X \cup (X \circledast A)$

Morphological boundary $\beta_A(X) = X \backslash (X \ominus A)$

Morphological reconstruction Marker F, mask G, structuring element A:
$X_0 = F$, $X_k = (X_{k-1} \oplus A) \cap G, \quad k = 1, 2, 3, \ldots$

Morphological skeleton Image X, structuring element A: $SK(X) = \bigcup_{n=0}^{N} S_n(X)$,
$S_n(X) = X \underset{n}{\ominus} A \setminus (X \underset{n}{\ominus} A) \circ A$, where $X \underset{0}{\ominus} A = X$ and N is the largest integer such that $S_N(X) \neq \emptyset$

Grey value dilation $(f \oplus b)(x, y) = \max_{(s,t)\in B} [f(x - s, y - t) + b(s, t)]$

Grey value erosion $(f \ominus b)(x, y) = \min_{(s,t)\in B} [f(x + s, y + t) - b(s, t)]$

Grey value opening $f \circ b = (f \ominus b) \oplus b$

Grey value closing $f \bullet b = (f \oplus b) \ominus b$

Morphological gradient $g = (f \oplus b) - (f \ominus b)$

Top-hat filter $T_{\text{hat}} = f - (f \circ b)$, **Bottom-hat filter** $B_{\text{hat}} = (f \bullet b) - f$

Sampling of continuous function $f(t)$: $\tilde{f}(t) = f(t)\, s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\,\delta(t - n\Delta T)$.
Fourier transform of sampled function: $\tilde{F}(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\Delta T})$

Sampling theorem Signal $f(t)$, bandwidth $\mu_{\max}$: If $\frac{1}{\Delta T} \geq 2\mu_{\max}$, $f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T)\,\text{sinc}\left[\frac{t-n\Delta T}{\Delta T}\right]$.

Sampling: downsampling by a factor of 2: $\downarrow_2 (a_0, a_1, a_2, \ldots, a_{2N-1}) = (a_0, a_2, a_4, \ldots, a_{2N-2})$

Sampling: upsampling by a factor of 2: $\uparrow_2 (a_0, a_1, a_2, \ldots, a_{N-1}) = (a_0, 0, a_1, 0, a_2, 0, \ldots, a_{N-1}, 0)$

Set, circularity ratio $R_c = \frac{4\pi A}{P^2}$ of set with area A, perimeter P

Set, diameter $\text{Diam}(B) = \max_{i,j} [D(p_i, p_j)]$ with p_i, p_j on the boundary B and D a distance measure

Sinc function $\text{sinc}\,(x) = \frac{\sin(\pi x)}{\pi x}$ when $x \neq 0$, and $\text{sinc}\,(0) = 1$. If $f(t) = A$ for $-W/2 \leq t \leq W/2$ and zero elswhere (block signal), then its Fourier transform is $F(\mu) = A\,W \text{sinc}\,(\mu\, W)$

Spatial moments of an $M \times N$ image $f(x, y)$: $m_{pq} = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} x^p\, y^q\, f(x, y), \quad p, q = 0, 1, 2, \ldots$

Statistical moments of distribution $p(i)$: $\mu_n = \sum_{i=0}^{L-1}(i - m)^n\, p(i)$, $m = \sum_{i=0}^{L-1} i\, p(i)$

Signal-to-noise ratio, mean-square $\text{SNR}_{\text{rms}} = \frac{\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\hat{f}(x,y)^2}{\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left(\hat{f}(x,y)-f(x,y)\right)^2}$

Wavelet decomposition with scaling function h_ϕ, wavelet function h_ψ. For $j = 1, \ldots, J$:
Approximation: $c_j = \mathbf{H}c_{j-1} =\downarrow_2 (h_\phi * c_{j-1})$; Detail: $d_j = \mathbf{G}c_{j-1} =\downarrow_2 (h_\psi * c_{j-1})$

Wavelet reconstruction with dual scaling function $\tilde{h}_\phi$, dual wavelet function $\tilde{h}_\psi$. For $j = J, J-1, \ldots, 1$:
$c_{j-1} = \tilde{h}_\phi * (\uparrow_2 c_j) + \tilde{h}_\psi * (\uparrow_2 d_j)$

Wavelet, Haar basis $h_\phi = \frac{1}{\sqrt{2}}(1, 1)$, $h_\psi = \frac{1}{\sqrt{2}}(1, -1)$, $\tilde{h}_\phi = \frac{1}{\sqrt{2}}(1, 1)$, $\tilde{h}_\psi = \frac{1}{\sqrt{2}}(1, -1)$

Answers

Problem 1 (30 pt)

a. An example for which the filter extracts horizontal edges:

-1	-2	-1
0	0	0
1	2	1

b. An example for which the filter smooths the input image: (× $\frac{1}{9}$)

1	1	1
1	1	1
1	1	1

c. An example of a discrete Laplacian:

0	-1	0
-1	4	-1
0	-1	0

d. An example for which the result can be also achieved by two passes using 1-D masks is the Sobel mask:

-1	-2	-1
0	0	0
+1	+2	+1

This can be obtained as the sequence of the following two 1-D masks: $\begin{bmatrix}+1 & +2 & +1\end{bmatrix}$ and $\begin{bmatrix}-1\\0\\+1\end{bmatrix}$

e. The average grey value of the filtered image is equal to the average grey value of the input image when $\sum_i w_i = 1$.

f. The average grey value of the filtered image is equal to zero when $\sum_i w_i = 0$.

Problem 2 (30 pt)

a. The marker image F needs to have at least one 1-pixel of every component which must be extracted. So we take F as an image of the same size as the input which is everywhere zero, except at the boundary, where it has a 1 for each pixel where the input also has a 1.
The mask image G is equal to the input image.
The structuring element B can be either

0	1	0
1	1	1
0	1	0

or

1	1	1
1	1	1
1	1	1

depending on the connectivity (the origin is at the center pixel).

b. The iteration is:

$$X_0 = F$$
$$X_k = (X_{k-1} \oplus B) \cap G, \quad k = 1, 2, 3, \ldots$$

So in each step, the current image X_{k-1} is dilated by B which expands it; the intersection with G ensures that X_k is never larger than the input image. After termination, all components touching the boundary have been extracted.

c. To get all letters of the input image except those which touch the boundary, we simply take the set difference of the input image and the output of step 1.

d. Image processing tasks involving morphological reconstruction: Hole filling; boundary extraction; extraction of connected components; convex hull computation.

Problem 3 (30 pt)

Here many answers are possible. Let us label the textures by:

a	b	c
d	e	f

a. Image operation / descriptors: mean grey level; variance of grey level; Fourier coefficients; co-occurrence matrix; granulometry.

b. Texture (e) has a mean grey level which is much larger than that of the other textures. This can be used to separate it from the others. So (a-d) and (f) remain. Textures (a) and (f) contain periodic structures. Looking for Fourier spectra with large amplitudes separates these two from (b-d). Texture (a) can be distinguished from (f) since its periodicities in the horizontal and vertical direction are very different, while for (f) they are the same. To distinguish (b), (c) and (d) we can first compute a granulometry: (b) has the smallest grains, those of (c) are intermediate and (d) has the largest grains.